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THE USE OF THE PREDICTIVE-MODEL METHOD TO SOLVE INVERSE PROBLEMS OF GAS DYNAMICS†

A. A. KOSTOGLOTOV and V. N. TARAN

Rostov-on-Don

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A new approach to the solution of inverse problems of gas dynamics is considered, based on the use of the predictive-model method [1] using an algorithm with Green's matrix, described earlier in [2] for the case of finite-dimensional control systems. This approach enables one to obtain an analytic expression for the generalized external forces, which considerably simplifies the procedure for finding a solution of the inverse problem. An example for the system of equations of one-dimensional gas dynamics is given.

1. FORMULATION OF THE PROBLEM

Consider the system of equations of one-dimensional gas dynamics in the region $\overline{Q}=Q+\partial Q\in R^2$, $\overline{Q}=\{t,\ r: t\in[0,\ T],\ r\in[0,\ R(t)]\}$ in the form of a vector differential equation in normal form

$$\partial \mathbf{w}/\partial t+\mathbf{F}(\mathbf{w},\mathbf{w}_r)=\mathbf{f}(t,r)\quad (\mathbf{w}_r=\partial \mathbf{w}/\partial r) \tag{1.1}$$

with initial and boundary conditions

$$\mathbf{w}(0,r)=\mathbf{w}^0(r),\quad S(w(t,0),w(t,R),r,t)=0 \tag{1.2}$$

Here $\mathbf{w}(t,r)\in L_3^2(\overline{Q})$ is the vector of the gas-dynamic parameters, $\mathbf{F}(\mathbf{w},\ \mathbf{w}_r)\in C^2$ is a vector-function which satisfies the conditions of hyperbolicity [3]: (1) all the eigenvalues of the matrix $\mathbf{A}=\partial\mathbf{F}/\partial\mathbf{w}_r$ are real, and (2) there is a basis in the space E_3, consisting of the left eigenvectors of the matrix A, $\mathbf{f}(t,\ r)\in L_3^2(Q)$ is a vector function which has the meaning of the generalized external forces, and $S(\mathbf{w}(t,0),\ \mathbf{w}(t,R),\ r,\ t)\in C^2$ is a vector function which ensures that the initial and boundary conditions are matched: $S(\mathbf{w}^0(0),\ \mathbf{w}^0(R),\ r,\ 0)=0$. We will assume that $f(t,\ r)$ uniquely defines the generalized solution of problem (1.1), (1.2).

We will assume that we are given the programme of the motion of the system considered in the form of the state vector $\varphi(t,r)\in L_3^2(\overline{Q})$.

The problem consists of determining the vector $f_0(t,\ r)$ for which the programmed motion $\varphi(t,\ r)$ is one of the possible motions of system (1.1), (1.2).

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2. METHOD OF SOLUTION

According to Hamilton's principle of least action we can write the functional that is stationary along the trajectory of the programmed motion $\varphi(t, r)$ of system (1.1), (1.2) in the form

$$J_0[\mathbf{w}] = \| \mathbf{w}(t,r) - \varphi(t,r) \| + \tfrac{1}{2}(\mathbf{K}^{-1}\mathbf{f},\mathbf{f}) \tag{2.1}$$

where $\mathbf{K}$ is a certain positive-definite matrix of quadratic form, and $\|\cdot\|$ and $(\cdot,\cdot)$ are the norm and scalar product in $L_3^2(\overline{Q})$, respectively.

The problem of finding $f_0(t, r)$ can then be treated as the problem of minimizing the functional (2.1)

$$J_0[\mathbf{w}] \to \inf_{\mathbf{f}}, \quad \mathbf{w} \in L_3^2(\overline{Q}), \quad \mathbf{f} \in L_3^2(Q)$$

with constraints specified in the form (1.1), (1.2).

It has been shown (see, for example, [4]), that the complexity of the problem can be reduced considerably by minimizing the functional of generalized work

$$J[\mathbf{w}] = \| \mathbf{w}(t,r) - \varphi(t,r) \| + \frac{1}{2}(\mathbf{K}^{-1}\mathbf{f},\mathbf{f}) + \frac{1}{2}(\mathbf{K}^{-1}\mathbf{f}_0,\mathbf{f}_0) =$$

$$= \int_0^T H[t,\mathbf{w}]dt + \frac{1}{2}\int_0^T \int_0^R (\mathbf{f}^T\mathbf{K}^{-1}\mathbf{f} + \mathbf{f}_0^T\mathbf{K}^{-1}\mathbf{f}_0)dr\,dt \tag{2.2}$$

$$\left(H[t,\mathbf{w}] = \int_0^R (\mathbf{w}(t,r) - \varphi(t,r))^2 dr \right)$$

Note that the case when $\|\mathbf{K}^{-1}\| = 0$ is of no interest since there is no guarantee here of the uniqueness of the element $\mathbf{f}_0 \in L_3^2(Q)$ [5], which contradicts the physical meaning of the problem.

Consider the Lyapunov functional of the form

$$V[t,\mathbf{w}] = \int_t^T \int_0^R [\mathbf{w}(s,r) - \varphi(s,r)]^2 dr\,ds + \frac{1}{2}\int_t^T \int_0^R \mathbf{f}^T(s,r)\mathbf{K}^{-1}\mathbf{f}(s,r)dr\,ds$$

We isolate an arbitrary internal point $\zeta(t, r)$ of the region Q and the region Q_ε of small measure $\text{mes}Q_\varepsilon = m_\varepsilon > 0$, lying completely inside Q and containing ζ as an internal point. We specify the variation of the function $\mathbf{w}(t, r)$ by the following formula [6]

$$\delta\mathbf{w}(t,r) = \begin{cases} C > 0, & (t,r) \in Q_\varepsilon \\ 0, & (t,r) \notin Q_\varepsilon \end{cases}$$

We define the functional derivative for the functional as follows [6]:

$$\frac{\delta I[\mathbf{w}]}{\delta \mathbf{w}} = \lim_{\|\delta\mathbf{w}\|_{L_3^2} \to 0} \frac{I[\mathbf{w} + \delta\mathbf{w}] - I[\mathbf{w}]}{\|\delta\mathbf{w}\|_{L_3^2}}$$

Theorem. For process (1.1), (1.2) and the programmed motion $\varphi(t, r)$ an extremum of the functional (2.2) is reached when

$$\mathbf{f}_0(t,r) = -\mathbf{K}\int_t^T\int_0^R \mathbf{G}^*(t,s,r,\xi)\frac{\delta H[s, y(s,\xi,\mathbf{w}(t,r))]}{\delta\mathbf{w}}d\xi\,ds$$

where $\mathbf{y}(s, \xi\, \mathbf{w}(t, r))$ is the free motion ($\mathbf{f}(s, \xi) \equiv 0$) of system (1.1), (1.2) in the section $s \in [t, T]$, which depends on the current state ($\mathbf{w}(t, r)$, and $\mathbf{G}^*(t, s, r, \xi)$ is Green's matrix, conjugate with respect to the variables r and ξ), of the linearized initial boundary-value problem (1.1), (1.2)

$$\frac{\partial \mathbf{G}}{\partial t} - \frac{\partial \mathbf{F}}{\partial \mathbf{w}}\mathbf{G} - \frac{\partial \mathbf{F}}{\partial \mathbf{w}_r}\frac{\partial \mathbf{G}}{\partial r} = \delta(t-s)\delta(r,\xi)$$

$$\mathbf{G}(t,s,0,\xi) = \mathbf{G}(t,s,R(t),\xi) = 0$$

and the derivatives $\partial\mathbf{F}/\partial\mathbf{w}$, $\partial\mathbf{F}/\partial\mathbf{w}_r$ are calculated on the trajectories of the free motion $\mathbf{y}(s, \xi, \mathbf{w}(t, r))$.

Proof. By the method of dynamic programming, a minimum of the functional (2.2) is reached on the solution of the functional equation

$$\min_{\mathbf{f}} \left\{ \frac{dV}{dt} + H + \frac{1}{2}\int_0^R [\mathbf{f}^T\mathbf{K}^{-1}\mathbf{f} + \mathbf{f}_0^T\mathbf{K}^{-1}\mathbf{f}_0]dr \right\} = 0 \tag{2.3}$$

written for the Lyapunov functional $V[t, \mathbf{w}]$, which depends parametrically on time and is defined on the set of functions $\mathbf{w}(t, r)$ which satisfying system (1.1), (1.2).

We can represent the derivative dV/dt in the region Q, taking (1.1) into account, in the form [6]

$$\frac{dV[t,\mathbf{w}]}{dt} = \frac{\partial V[t,\mathbf{w}]}{\partial t} + \int_0^R \frac{\delta V[t,\mathbf{w}]}{\delta \mathbf{w}}(-\mathbf{F}+\mathbf{f})dr$$

The solution of the variational problem in question is then the function

$$\mathbf{f}_0(t,r) = -\mathbf{K}\,\delta V[t,\mathbf{w}]/\delta\mathbf{w} \tag{2.4}$$

where V is the solution of the linear functional equation

$$\frac{dV[t,\mathbf{w}]}{dt} - \int_0^R \frac{\delta V[t,\mathbf{w}]}{\delta \mathbf{w}}\mathbf{F}(\mathbf{w},\mathbf{w}_r)dr = -H[t,\mathbf{w}] \tag{2.5}$$

This can be checked (see, for example, [4]) by substituting the expression for dV/dt into (2.3) and subsequent minimization with respect to $\mathbf{f}$ using (2.5). It has to be said that in expression (2.4) the argument $\mathbf{f}_0(t, r)$ contains $\mathbf{w}(t, r)$—the free element of phase space and, consequently, this expression must be understood as the inverse relationship, and not programmed control, which minimizes the specified functional.

Note that Eq. (2.5) is related to the equation of the characteristic

$$\partial\mathbf{w}/\partial t = -\mathbf{F}(\mathbf{w},\mathbf{w}_r) \tag{2.6}$$

for which $dV/dt = -H$ and, correspondingly.

$$V[t,\mathbf{y}(t,r)] = \int_t^T H[s,\mathbf{y}(t,s,r,\mathbf{w}(t,r))]ds \tag{2.7}$$

In the last equation, y is the phase trajectory of system (2.6) with the initial conditions $\mathbf{y}(t, s, r)|_{s=t} = \mathbf{w}(t, r)$.

Consider the ε-neighbourhood of $\mathbf{y}(t, s, r)|_{s=t}$ in the space $L_3^2(Q)$. For a limited variation of the initial conditions $\mathbf{w}(t, r) + \delta\mathbf{w}(t, r)\|_{L_3^2(Q)} \leqslant \varepsilon$ we will denote the variation of the solution of Eq. (2.6) by

$$\mathbf{g}(t,s,r) = \mathbf{y}(t,s,r,\mathbf{w}+\delta\mathbf{w}) - \mathbf{y}(t,s,r,\mathbf{w})$$

In view of (2.7) we have

$$\delta V[t,y]=\int_t^T\int_0^R \frac{\delta H}{\delta w}[s,\mathbf{y}(t,s,r,\mathbf{w}(t,r))]\,\mathbf{g}(t,s,r)dr\,ds$$

where $\mathbf{g}(t,s,r)$ is the solution of the homogeneous initial boundary-value problem

$$\frac{\partial \mathbf{g}}{\partial s}-\frac{\partial \mathbf{F}}{\partial \mathbf{w}_r}\frac{\partial \mathbf{g}}{\partial r}-\frac{\partial \mathbf{F}}{\partial \mathbf{w}}\mathbf{g}=0$$

$$\mathbf{g}(t,s,r)|_{s=t}=\delta\mathbf{w}(t,r),\qquad \mathbf{g}(t,s,o)=\mathbf{g}(t,s,R)=0$$

This solution, by virtue of the linearity of the boundary-value problem, can be represented in the form ($\mathbf{G}(t,s,r,\xi)$ is Green's matrix)

$$\mathbf{g}(t,s,r)=\int_0^R \mathbf{G}(t,s,r,\xi)\delta\mathbf{w}(t,\xi)d\xi \tag{2.8}$$

Then, we have for the variation of the functional V

$$\delta V[t,\mathbf{y}]=\int_0^R\left\{\int_t^T\int_0^R \mathbf{G}^*(t,s,r,\xi)\frac{\delta H}{\delta \mathbf{w}}[s,\mathbf{y}(t,s,\xi,\mathbf{w}(t,r))]d\xi\,ds\right\}\delta\mathbf{w}(t,r)dr \tag{2.9}$$

where $\mathbf{G}^*$ is a function conjugate to $\mathbf{G}$ with respect to the variables r and ξ, and $\mathbf{G}$ and $\delta H/\delta\mathbf{w}$ are on the trajectory of free motion [2] of system (2.6).

Using expression (2.9) to calculate $\mathbf{f}_0$ from (2.4), we obtain

$$\mathbf{f}_0(t,r)=-\mathbf{K}\int_t^T\int_0^R \mathbf{G}^*(t,s,r,\xi)\frac{\delta H[s,\mathbf{y}(s,\xi,\mathbf{w}(t,r))]}{\delta\mathbf{w}}d\xi\,ds \tag{2.10}$$

which proves the theorem.

Expression (2.10) enables us to construct a vector-function $\mathbf{f}_0$ as a solution of the inverse problem (1.1), (1.2) in the sense of a minimum of the functional of generalized work (2.2).

The explicit form of the solution obtained enables us to reduce considerably the work involved in obtaining the solution of the inverse problem, and in certain cases enables us to find a solution when traditional methods give no result due to the considerably complexity and length of the analysis.

3. EXAMPLE

Suppose the system of equations of gas dynamics describes the expansion of a spherical volume in a medium with a black pressure. The corresponding equations in spherical coordinates have the form

$$\begin{aligned}
&\frac{\partial \upsilon}{\partial t}+\upsilon\frac{\partial \upsilon}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=f_1(t,r)\\
&\frac{\partial \rho}{\partial t}+\upsilon\frac{\partial \rho}{\partial r}+\rho\frac{\partial \upsilon}{\partial r}+\frac{2\rho\upsilon}{r}=f_2(t,r)\\
&\frac{\partial p}{\partial r}+\upsilon\frac{\partial p}{\partial r}+\gamma p\frac{\partial \upsilon}{\partial r}+\frac{2\gamma p\upsilon}{r}=f_3(t,r)
\end{aligned} \tag{3.1}$$

where υ, ρ and p are the gas-dynamic parameters of the perturbed medium, $\gamma=1.4$ is the adiabatic index, and f_1, f_2 and f_3 are sinks or sources of mass, momentum and energy, respectively, defined in the region of smooth flow.

The boundary conditions are specified at the centre $r=0$

$$\upsilon(t,0)=0 \tag{3.2}$$

and on the front $r=r(t)$ of the shock wave, which propagates into the unperturbed medium ($\upsilon=\upsilon_1=0$, $\rho=\rho_1=\text{const}$, $p=p_1=\text{const}$)with a velocity $D=dR/dt$. The dimensionless forms [7] of the parameters on the shock-wave front have the form

$$p_R=\frac{1}{4}\left[(\gamma+1)\upsilon_R^2+\upsilon_R\sqrt{(\gamma+1)\upsilon_R^2+16\gamma}\right],\quad \rho_R=\frac{(\gamma+1)p_R+\gamma-1}{\gamma+1+(\gamma-1)p_R}$$
$$D=\{[\gamma+1)p_R+\gamma-1]/2\}^{1/2}. \tag{3.3}$$

The initial conditions $\upsilon(0,\ r)=\upsilon^0(r)$, $\rho(0,\ r)=\rho^0(r)$, $p(0,\ r)=p^0(r)$, $r\in[0,R(t)]$ represent the self-similar solution of the problem of a point explosion [7].

The programmed motion is specified on the shock-wave front when $t\in[t_0,\ t_0+T]$

$$\varphi(t,R)=\varphi_R(t)=(\upsilon_R^*(t),\ \rho_R^*(t),\ p_R^*(t))^{\text{T}} \tag{3.4}$$

For system (3.1), the boundary conditions (3.2) and (3.3) and the programmed motion (3.4), we will seek a solution of the inverse problem $\mathbf{f}_0=(f_{01},\ f_{02},\ f_{03})$ from the condition for a minimum of the functional

$$J=\frac{1}{2}\int_0^T[\mathbf{w}(t)-\varphi_R(t)]^2dt+\frac{1}{2}\int_0^T\int_0^R(\mathbf{f}^{\text{T}}\mathbf{K}^{-1}\mathbf{f}+\mathbf{f}_0^{\text{T}}\mathbf{K}^{-1}\mathbf{f}_0)dr\,dt \tag{3.5}$$
$$(\mathbf{w}(t)\ =\ (\upsilon_R(t),\ \rho_R(t),\ p_R(t))^{\text{T}})$$

Then, by the statement of the theorem

$$\mathbf{f}_0(t,r)=-\mathbf{K}\int_t^T\int_0^R\mathbf{G}^*(t,s,r,\xi)[\mathbf{y}(s,R,\mathbf{w}(t,r))-\varphi_R(s)]d\xi ds$$

where $\mathbf{G}$ is Green's matrix of the initial boundary-value problem

$$\frac{\partial\mathbf{G}}{\partial t}+\mathbf{A}\frac{\partial\mathbf{G}}{\partial r}+\mathbf{BG}=\delta(t-s)\delta(r,\xi),\quad \mathbf{G}(t,s,0,\xi)\ =\ \mathbf{G}(t,s,R(t),\xi)\ =\ 0$$

where

$$\mathbf{A}=\begin{Vmatrix}\upsilon & 0 & 1/p\\ \rho & \upsilon & 0\\ \gamma p & 0 & \upsilon\end{Vmatrix},\quad \mathbf{B}=\begin{Vmatrix}\dfrac{\partial\upsilon}{\partial r} & -\dfrac{1}{\rho^2}\dfrac{\partial p}{\partial r} & 0\\ \dfrac{\partial\rho}{\partial r}+\dfrac{\rho}{r} & \dfrac{\partial\upsilon}{\partial r}+\dfrac{2\upsilon}{r} & 0\\ \dfrac{\partial p}{\partial r}+\gamma\dfrac{p}{r} & 0 & \gamma\dfrac{\partial\upsilon}{\partial r}+\dfrac{2\gamma\upsilon}{r}\end{Vmatrix}$$

are calculated on the free motion $\mathbf{y}(s,\ \xi,\ \mathbf{w}(t,\ r))$ of system (3.1)–(3.3).

In the computational algorithm the number of spatial coordinate nodes was taken to be 15. The distributed gas-dynamic parameters were approximated by the spline-function method. The Courant number was chosen, from considerations of computational stability, to be equal to 0.75 for a value of the step in dimensionless time $\tau=10^{-6}$. The sliding interval of the calculation $T=5\tau$. The overall time required for the calculation is 30τ. The coefficients of the diagonal matrix K were chosen experimentally from the condition for the programmed motion to be identical with the solution of problem (3.1)–(3.3) and was taken to be $k_{ii} = 10^{-10}$. The time of the beginning of the calculations $t_0/t_g = 5.455\times10^{-4}$ and the dimensionless parameters on the front in this case $R(t_0)/R_g = 4.942\times10^{-2}$ were determined by the following values

$$\frac{\upsilon_R^0}{c} = 30.55; \quad \frac{p_R^0}{p_1} = 1122; \quad \frac{\rho_R^0}{\rho_1} = 5.969$$

Here t_g and R_g are the dynamic time and the length, respectively [7], c is the velocity of sound in the unperturbed medium, and ρ_1 and p_1 are the density and pressure of the medium under normal conditions.

The results of calculations for the time interval considered can be approximated by linear relations

$$\frac{\upsilon_R^0 - \upsilon_R}{c} = k_\upsilon t; \quad \frac{p_R^0 - p_R}{p_1} = k_p t; \quad \frac{\rho_R^0 - \rho_R}{\rho_1} = k_\rho t$$

where $k_\upsilon = 2.86\times10^5$, $k_p = 2.03\times10^6$ and $k_\rho = 6.0$ for the unperturbed motion, $k_\upsilon = 3.83\times10^5$, $k_p = 2.63\times10^6$ and $k_\rho = 86.6$ for the programmed motion, and $k_\upsilon = 3.83\times10^5$, $k_p = 2.70\times10^6$ and $k_\rho = 83.3$ for the solution of problem (3.1)–(3.3), (3.5) obtained using the proposed approach. Figures 1–3 show graphs of the functions f_{01}, f_{02} and f_{03}, which are the solution of the inverse problem in question.

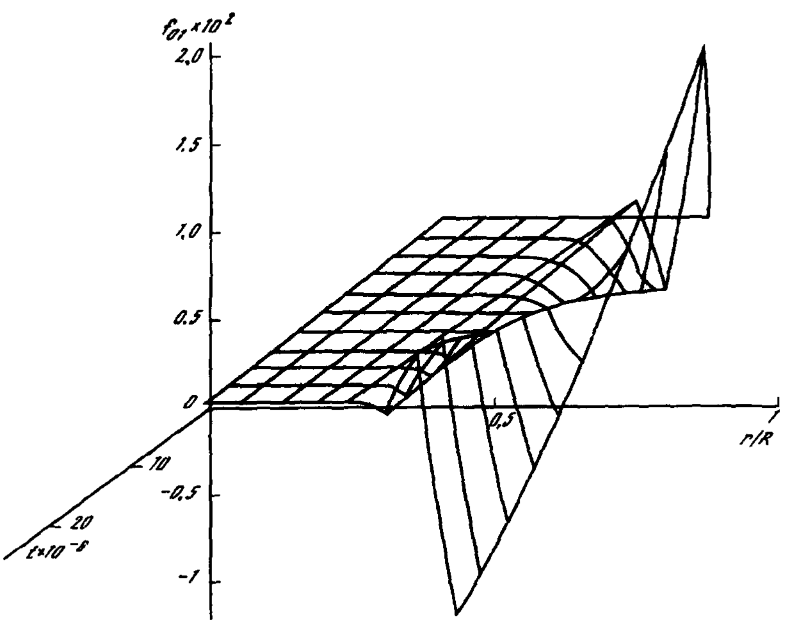

Fig. 1.

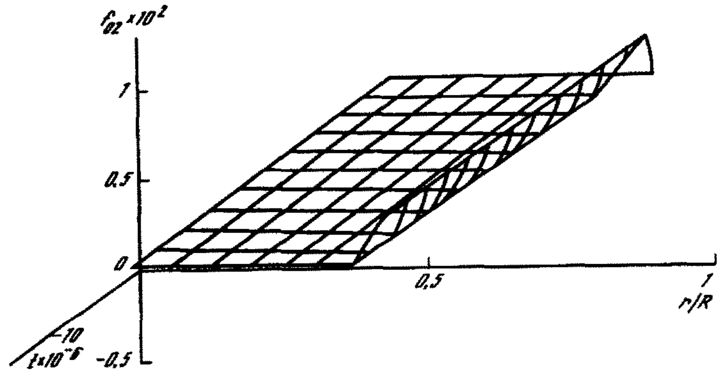

Fig. 2.

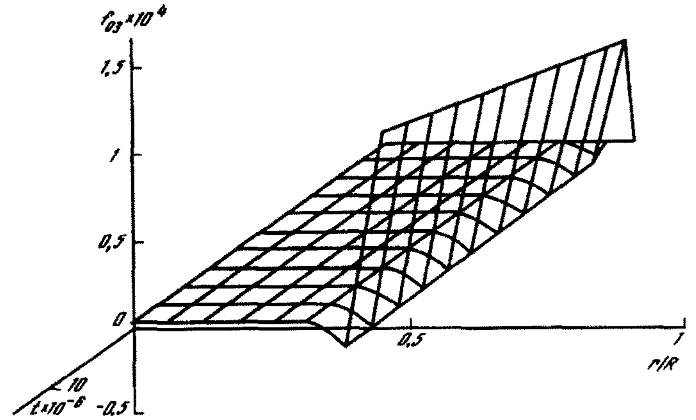

Fig. 3.

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